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1D Schrödinger equations with Coulomb-type potentials

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Abstract. We employ Laplace and Fourier transforms in momentum space to find the bound states of the 1D Schrödinger equations with two different potentials; $1/x$ and $1/|x|$.

By performing inverse transforms we show that for the potential $1/|x|$ the solutions in real space reduce to those of the 1D hydrogen atom with eigenenergies proportional to $1/n^2$ with n integer. Analogously, we find that for the potential $1/x$ the eigenenergies are proportional to $1/(n+\frac{1}{2})^2$ and the eigenfunctions can be expressed in terms of fractional derivatives. Taking into account that both potentials are singular (the $1/x$ potential is analytical and the $1/|x|$ potential is not), we analyse the nature of their bound states.

1. Introduction

The stationary states of the one-dimensional (1D) Schrödinger equation with a $1/|x|$ potential describing the 1D hydrogen atom (1D H atom) have attracted a great deal of interest [1–6]. This equation is related to the exciton problem in the study of high temperature superconductors [7], semiconductor quantum wires† [8–11], polymers [12, 13], and also due to the existence of image forces on 1D electron gas at the helium surface, to the Wigner crystal [14, 15]. In contrast to the r-dependence of the 3D H atom eigenstates, the 1D H atom states have parity since the eigenfunctions are also defined for negative values of the argument x. In fact, besides a factor r, the 3D H atom states are equal to those of the s-states ($l = 0$) of the 1D H atom for positive x. Flugge and Marschall [1] concluded that only the odd states were bound solutions of the 1D H atom, while Loudon [2] claimed that the even states were solutions as well (including a ground state with infinite binding energy), being degenerate with the odd states. Andrews [3] objected to the existence of Loudon's 'ground state' and Xianxi *et al* [5] claimed (apparently with conclusive arguments) that only eigenstates with even parity exist. The latter authors pointed out, in connection with Loudon's ground state, that for some systems singular states have to be taken into account, such as the famous H atom ground state of the Dirac equation. Besides confirming a discrete negative-energy spectrum corresponding to odd-parity solutions, Heines and Roberts [16] discovered a continuous spectrum of even states that is unbounded from below, which Andrews [17] correctly observed does not form an orthogonal set (that is, it is not rectifiable using the Schmidt orthogonalization procedure). An excellent and detailed account of the 1D H atom is found in [5, 18].

A related 1D Schrödinger equation with a Coulomb-type $1/x$ potential, seems not to have been investigated before. However, this potential could be used to describe asymmetric wells in 1D systems. For example, since in a semiconductor or insulator $1/\epsilon x$ is the screened potential

† For experimental and theoretical reviews on quantum wire systems see [8].

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and the dielectric function ϵ can be negative for some frequencies, then we could think of an impurity at the boundary of two 1D systems characterized by dielectric functions ϵ_1 and ϵ_2 where $\epsilon_1 = -\epsilon_2$. As will be shown here later, both potentials $1/x$ and $1/|x|$ will be treated in a similar mathematical fashion.

The outline of the paper is the following. In order to gain insight on the nature of the less known 1D $1/x$ potential we solve this problem both classically and semiclassically in section 2. In section 3 we solve the 1D Schrödinger equations with potentials $1/x$ and $1/|x|$ in momentum space to obtain in a very simple way both their eigenenergies and eigenfunctions. This procedure seems to be simpler than Frobenius series methods [5] or generalized Laplace transforms involving generalized functions [18] to solve the 1D H atom. Section 4 is devoted to find the wavefunctions in real space by performing the inverse transforms. Finally, in section 5 we explore the continuum-spectrum solutions of the $1/x$ potential and we summarize our results.

2. Classical and semiclassical analysis

Here we shall briefly analyse the classical and semiclassical behaviour of the 1D $1/x$ potential, together with the results of the 1D potential $1/|x|$ of Gordeyev *et al* [18]. First, we should mention that in contrast to $1/|x|$ (e.g. $x = 0$ is not just a pole), the 1D $1/x$ potential is an analytical function of x and thus its associated force does not contain any generalized function as it does for $1/|x|$. For bounded trajectories we set the total energy E to be negative and ($E = -|E|$);

$$-|E| = \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{Ze^2}{x}$$

hence

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\frac{2}{m}\left(\frac{Ze^2}{x} - |E|\right)}.$$

Classical motion is only allowed from the origin to the turning point where $\mathrm{d}x/\mathrm{d}t = 0$, that is, in the range $0 < x < Ze^2/|E|$, which implies that motion for this potential is exactly the same as the corresponding 3D Kepler problem with null angular momentum. A parametric description of this movement is usually expressed in terms of the function

$$x(\xi) = 2a\sin^2(\xi/2)$$

where $a = Ze^2/2|E|$ and ξ is given by $\omega t = \xi - \sin\xi$ with $\omega = \sqrt{Ze^2/ma^2}$. The explicit time dependent expression for the solution is found by expanding $x(\xi)$ in Fourier series and is given by [19]

$$x(t) = 2a\left(-\frac{3}{4} + \sum_{n=1}^{\infty} J'_n(n)\frac{\cos(n\omega t)}{n}\right)$$

where $J'_n(x)$ is the derivative of the Bessel function of order n. The features of this motion are those of the 3D Kepler system, namely the amplitude of oscillation and the period of motion are the same. Analogously to trajectories in the 1D $1/|x|$ potential [18], $x(t)$ in our case is continuous but in contrast, it is not differentiable at the origin even in the case of treating it as a generalized function; as is the case for the trajectory in the $1/|x|$ potential [18]. This can be seen by noting that $\mathrm{d}x/\mathrm{d}t \to \pm\infty$ as $x^+ \to 0$, where the negative infinite symbol describes the particle speed when it reaches the origin and the positive one when it leaves this point. This change of momentum is provided to the system by the classically impenetrable barrier at the origin. This divergence in the particle speed obviously suggests that this classical problem should be solved relativistically.

The semiclassical analysis for the $1/x$ potential can be performed by using the well known Bohr–Sommerfeld quantization condition

$$2\pi n = \oint p\,\mathrm{d}x/\hbar = 2\int_0^{\zeta_r} \sqrt{1-\gamma/\zeta}\,\mathrm{d}\zeta \tag{1}$$

where $\gamma = -(Ze^2/\hbar E)\sqrt{-2m/E}$, $\zeta = \sqrt{-2mE}x/\hbar$ and the return point $\zeta_r = -Ze^2/E > 0$. This yields the following energy levels

$$E_n = -\frac{mZ^2e^4}{2\hbar^2n^2} \tag{2}$$

which coincidentally are those of the 3D H system. However, this treatment is not correct since the WKB approximation, on which equation (1) is based, does not apply near the origin because $\mathrm{d}\lambda/\mathrm{d}x\,|_{x\to 0} = \mathrm{d}(\hbar/p)/\mathrm{d}x\,|_{x\to 0} \sim x^{-1/2}$ is not small, but tends to infinity as $x \to 0$ (λ is the de Broglie wavelength). Thus, the well known WKB solution in a potential well

$$\psi = \frac{1}{\sqrt{p}} C \sin\left[(1/\hbar)\int_x^{x_r} p\,\mathrm{d}x + \pi/4\right] \qquad x < x_r \tag{3}$$

where $p = (2mZe^2/x - 2m|E|)^{1/2}$, only represents those points x which are not near the origin. Here C is a constant to be determined.

In the vicinity of the origin the correct approach is to approximate the Schrödinger equation (9) by

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} + \frac{\gamma}{\zeta}\psi = 0 \tag{4}$$

when $1 \ll |\gamma/\zeta|$ which is valid near the origin. It is straightforward to show that the general solution of this equation is given by

$$\psi = \begin{cases} A_1\sqrt{\zeta}J_1(2\sqrt{\gamma\zeta}) + A_2\sqrt{\zeta}N_1(2\sqrt{\gamma\zeta}) & \zeta > 0 \\ -A_1\sqrt{\zeta}I_1(2\sqrt{\gamma\zeta}) - A_2\sqrt{\zeta}K_1(2\sqrt{\gamma\zeta}) & \zeta < 0 \end{cases} \tag{5}$$

where $J_1(x)$, $N_1(x)$, $I_1(x)$ and $K_1(x)$ are the Bessel, Neumann and modified Bessel and Neumann functions of order one, respectively. A reasonable physical requirement is to ask ψ in equation (5) to vanish when $\zeta \to -\infty$, because classically there is not a particle for $\zeta < 0$, hence $A_1 = 0$. The semiclassical expression equation (3) is to be compared with the asymptotic form of equation (5) for the positive region, that is to say, for large values of its argument. On the one hand, equation (5) reduces to

$$\psi = A_2\zeta^{1/4}\cos(2\sqrt{\gamma\zeta} - \pi/4) \qquad \zeta \gg 1. \tag{6}$$

On the other hand, using the following approximation near the origin

$$\int_x^{x_r} p\,\mathrm{d}x \approx \int_0^{x_r} p\,\mathrm{d}x - \int_0^x \sqrt{2mZe^2/x}\,\mathrm{d}x$$

we can rewrite equation (3) as

$$\psi \approx -Cx^{1/4}\sin\left[2\sqrt{\gamma\zeta} - \frac{\pi mZe^2}{\hbar\sqrt{-2mE}} - \pi/4\right] \qquad x < x_r. \tag{7}$$

Finally, equations (6) and (7) describe the same wavefunction only if

$$E_n = -\frac{mZ^2e^4}{2\hbar^2(n+1/2)^2} \tag{8}$$

and $C = -A_2$ which is different from equation (2) and which just coincides for the higher levels where the WKB analysis is valid. Thus, surprisingly, WKB results given by equation (8) provide the exact result for the 1D $1/x$ potential as will be shown in the next section. That is, the WKB approach yields the exact eigenenergy results for both 3D H atom and for the $1/x$ potential.

Incidentally, for the 1D $1/|x|$ potential an antisymmetric function was constructed with the values of equation (5) for positive ζ by Gordeyev and Chhajlany [18] with $A_2 = 0$. By ignoring the Neumann function $\sqrt{\zeta}N_1(2\sqrt{\gamma\zeta})$ they did not include even solutions of the type found by Heines and Roberts [16], and therefore the negative energy continuum spectrum solutions [16] were lost.

We close this section by emphasizing that our treatment, which is beyond the usual semiclassical WKB one, predicts that a particle confined to the right-hand side of the $1/x$ potential can trespass the potential barrier at the origin since for negative values of x there exists an evanescent wavefunction given by $A_2\sqrt{\zeta}N_1(2\sqrt{\gamma\zeta})$. This is understandable since for $x^- \to 0$ the barrier becomes infinite but its thickness decreases, analogously to Dirac's delta potential.

3. Momentum space equation

The 1D Schrödinger equations with attractive Coulomb-like potentials $-Ze^2/x$ and $-Ze^2/|x|$ to be considered here are given by

$$\frac{d^2\psi_j}{d\zeta^2} + \left(\frac{\gamma}{f_j(\zeta)} - 1\right)\psi_j = 0 \tag{9}$$

where $\gamma = -(Ze^2/\hbar E)\sqrt{-2m/E}$, $\zeta = \sqrt{-2mE}x/\hbar$, $j = 1, 2$ and $f_1(\zeta) = \zeta$, $f_2(\zeta) = |\zeta|$. Here m and e are the mass and electric charge of the electron and Z is a positive integer. We restrict our work to consider just the bound states associated to these equations for which their corresponding wavefunctions are quadratic integrable in the whole space. For such states the Fourier transform is well defined, and thus we can take the Fourier transform of equation (9) to obtain

$$-(\bar{p}^2 + 1)\phi_j(\bar{p}) + \gamma \int_{-\infty}^{\infty} r_j(\bar{p} - \bar{p}')\phi_j(\bar{p}')\, d\bar{p}' = 0 \tag{10}$$

where we have used the convolution theorem. Here $r_j(\bar{p})$ is the Fourier transform of $1/f_j(\zeta)$ and $\phi_j(\bar{p})$ is the wavefunction in momentum space that is given by

$$\phi_j(\bar{p}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\ \psi_j(\zeta)e^{-i\bar{p}\zeta}. \tag{11}$$

If we consider first $f_1(\zeta) = \zeta$, then we have $r_1(\bar{p}) = -(i/2)\mathrm{sign}(\bar{p})$ [20] where $\mathrm{sign}(\bar{p}) = 1$ for $\bar{p} > 0$ and $\mathrm{sign}(\bar{p}) = -1$ for $\bar{p} < 0$, and hence equation (10) can be written as

$$-(p^2 + 1)\frac{dG_1}{d\bar{p}} - \frac{i\gamma}{2}(2G_1(\bar{p}) - G_1(\infty) - G_1(-\infty)) = 0$$

where we have introduced $G_1(p) = \int_0^p \phi_1(\bar{p}')\, d\bar{p}'$. The general solution of this equation is given by

$$G_1(\bar{p}) = \frac{G_1(\infty) + G_1(-\infty)}{2} - A_1 e^{-i\gamma \arctan \bar{p}} \tag{12}$$

where A_1 is an arbitrary constant to be determined from the normalization condition. If we evaluate equation (12) in ∞ and $-\infty$, we arrive to $G_1(\infty) - G_1(-\infty) = 2A_1 e^{i\gamma\pi/2} =$

$-2A_1\mathrm{e}^{-\mathrm{i}\gamma\pi/2}$, which allow us to determine that the eigenenergies of the system are given by $E_{1,n} = -mZ^2e^4/(2\hbar^2(n+\frac{1}{2})^2)$ with n an integer. That is, summarizing

$$\phi_{1,n}(\bar{p}) = \frac{\mathrm{d}G_{1,n}}{\mathrm{d}\bar{p}} = \mathrm{i}(2n+1)A_1\frac{\mathrm{e}^{-\mathrm{i}(2n+1)\arctan\bar{p}}}{\bar{p}^2+1}. \tag{13}$$

It is important to emphasize that our straightforward procedure to solve the 1D $1/x$ potential automatically provides the quantization condition. However, on using $G_1(\bar{p})$ we implicitly introduce an additional restriction on the solutions which consists in requiring $G_1(\infty) - G_1(-\infty) = \int_{-\infty}^{\infty}\mathrm{d}\bar{p}\,\phi_1(\bar{p})$ to exist; which is a stronger condition than the usual quadratic integrable condition. We shall conduct further analysis in section 5 to verify that the full spectrum has been identified.

For the $1/|x|$ potential the form of $\phi_2(\bar{p})$ is expected to be similar to $\phi_1(\bar{p})$ but we cannot use equation (10) because $r_2(\bar{p}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\zeta\,\mathrm{e}^{-\mathrm{i}\bar{p}\zeta}/|\zeta|$ is divergent. This fact makes the integral transform procedure less direct, since as will be shown, we will have to match the Laplace transform wavefunction at the origin according to the parity. The Laplace transform $\mathcal{L}$ of equation (9) for both the positive and negative part of the real axis, defined by

$$\phi_2(s) = \mathcal{L}[\psi_2(x)] = \int_0^{\infty}\mathrm{d}x\;\psi_2(x)\mathrm{e}^{-sx} \tag{14}$$

can be employed due to the property [20] $\mathcal{L}[\psi_2(x)/x] = \int_{\bar{p}}^{\infty}\phi_2(s)\,\mathrm{d}s$ which is valid when $\lim_{x->0}\psi_2(x)/x$ is well defined. Here s is understood as $s = \mathrm{i}\bar{p}$. Taking the Laplace transform of equation (9) yields

$$(\bar{p}^2+1)\frac{\mathrm{d}G_2}{\mathrm{d}\bar{p}} + \gamma(G_2(p) - G_2(\infty)) - \frac{\mathrm{d}\psi}{\mathrm{d}\zeta}(0^+) = 0 \tag{15}$$

where $G_2(\bar{p}) = \mathrm{i}\int_0^{\bar{p}}\phi_2(\mathrm{i}s')\,\mathrm{d}s'$ and $\mathrm{d}\psi\,(0^+)/\mathrm{d}\zeta$ is the right-hand side limit of $\psi(\zeta)$ at $\zeta = 0$. By solving this equation for $G_2(\bar{p})$ and rewriting the resulting expression in terms of $\bar{p}$, we arrive at

$$G_2(\bar{p}) - G_2(\infty) + \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}(0^+) = -A_2\mathrm{e}^{-\mathrm{i}\gamma\arctan\bar{p}} \qquad \bar{p} > 0 \tag{16}$$

where A_2 is an arbitrary constant to be determined by normalization. Using the same procedure but for the negative part of the real axis, we obtain

$$G_2(\bar{p}) - G_2(-\infty) + \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}(0^-) = -A_2\mathrm{e}^{-\mathrm{i}\gamma\arctan\bar{p}} \qquad \bar{p} < 0. \tag{17}$$

Since $1/f_2(\zeta)$ is an even potential there must be even and odd eigenfunctions in the system. For even functions $\psi(\zeta)$ we have $\mathrm{d}\psi\,(\zeta)/\mathrm{d}\zeta = -\mathrm{d}\psi\,(-\zeta)/\mathrm{d}\zeta$ and $G_2(\bar{p}) = -G_2(-\bar{p})$. Thus, adding equations (16) and (17) yields

$$G_2(\bar{p}) = -A_2\mathrm{e}^{-\mathrm{i}\gamma\arctan\bar{p}} \qquad \text{any} \quad \bar{p}. \tag{18}$$

Note that in this case there is no restriction on the value of γ meaning that even eigenfunctions which have a continuum spectrum are exactly the same type of solutions found by Heines and Roberts [16]. However, as mentioned in the introduction, it has been proved that these solutions are not mutually orthogonal and are not rectifiable by using the Schmidt method [17].

On the other hand, since for odd eigenfunctions $\mathrm{d}\psi\,(\zeta)/\mathrm{d}\zeta = \mathrm{d}\psi\,(-\zeta)/\mathrm{d}\zeta$, evaluation of equations (16) and (17) at ∞ and $-\infty$, respectively, allows us to write $A_2\mathrm{e}^{\mathrm{i}\gamma\pi/2} = A_2\mathrm{e}^{-\mathrm{i}\gamma\pi/2}$, which yields the eigenenergy spectrum; $E_{2,n} = -mZ^2e^4/(2\hbar^2n^2)$ with n an integer. This is exactly the same result obtained by Loudon [2] and Flugge and Marschall [1] for the 1D H atom by using a series expansion in the real space. Also, both 1D H atom eigenenergies and eigenfunctions were found in [18] by employing a cumbersome generalized Laplace transform which leads to the use of generalized functions such as Dirac's delta function. Analogously to equation (13), we have that

$$\phi_{2,n}(\bar{p}) = \frac{\mathrm{d}G_{2,n}}{\mathrm{d}\bar{p}} = 2\mathrm{i}nA_2 \frac{\mathrm{e}^{-2\mathrm{i}n \arctan \bar{p}}}{\bar{p}^2 + 1}. \tag{19}$$

The corresponding probability densities associated to each n-eigenfunction can be calculated from $P_{j,n}(\bar{p}) = |\mathrm{d}G_{j,n}/\mathrm{d}\bar{p}|^2$ yielding $P_{j,n}(\bar{p}) = A'^2_{j,n}/(1 + \bar{p}^2)^2$. Notice that for a given n, $P_{j,n}(\bar{p})$ has the same analytical form than that of the only bound state of an attractive delta potential, namely, we could adjust the strength of the delta potential in such way that we could reproduce $P_{j,n}(\bar{p})$ for any n.

4. Wavefunctions in real space

Due to the fact that the only difference between $G_1(\bar{p})$ and $G_2(\bar{p})$ are additive constants, the value of γ_n and $f_j(\zeta)$, we can manipulate them simultaneously to get the inverse Fourier transform of $\mathrm{d}G_{j,n}/\mathrm{d}\bar{p}$, $i = 1, 2$ in order to obtain the eigenfunctions in real space $\psi_{j,n}(x)$:

$$\psi_{j,n}(\zeta) = -(A'_{j,n}\zeta) \int_{-\infty}^{\infty} \mathrm{d}\bar{p}\, \mathrm{e}^{-\mathrm{i}\gamma_n \arctan \bar{p}} \mathrm{e}^{\mathrm{i}\bar{p} f_j(\zeta)}. \tag{20}$$

Next, using the identity $\arctan(u) = -(\mathrm{i}/2)\ln[(1+\mathrm{i}u)/(1-\mathrm{i}u)]$, equation (20) turns out to be

$$\psi_{j,n}(\zeta) = -A'_{j,n}(-1)^{\gamma_n/2}\zeta \int_{-\infty}^{\infty} \mathrm{d}\bar{p} \left(\frac{\mathrm{i}\bar{p} - 1}{\mathrm{i}\bar{p} + 1}\right)^{\gamma_n/2} \mathrm{e}^{\mathrm{i}\bar{p} f_j(\zeta)} \tag{21}$$

and if we introduce the dimensionless complex variable $z = (\mathrm{i}\bar{p} + 1)/(\mathrm{i}\bar{p} - 1)$, equation (21) can be expressed as the contour integral in the complex unit circle $|z| < 1$ given by

$$\psi_{j,n}(x) = -\mathrm{i}A'_{j,n}\zeta\, \mathrm{e}^{-f_j(\zeta)} \oint_C \mathrm{d}z\, \frac{\mathrm{e}^{-2f_j(\zeta)z/(1-z)}}{z^{\gamma_n/2}(1-z)^2}. \tag{22}$$

For the 1D H atom case ($j = 2$) for which $\gamma_n = 2n$ and $f_2(\zeta) = |\zeta|$, the contour integral in equation (22) is nothing but the complex representation of the associated Laguerre polynomials [21] $2\pi L^1_{n-1}(|\zeta|)$, that is

$$\psi_{2,n}(\zeta) = 2\pi A'_{2,n}\zeta\, \mathrm{e}^{-|\zeta|} L^1_{n-1}(2|\zeta|) \tag{23}$$

which is the same result found by Flugge and Marschall [1] so these eigenfunctions are just the odd ones in agreement with Xianxi *et al* [5].

For the $1/x$ potential ($j = 1$) for which $\gamma_n = 2n+1$, it is convenient to rewrite the contour integral of equation (22) in terms of the variable $s - \zeta = \zeta z/(1-z)$ to obtain

$$\frac{\mathrm{e}^{2\zeta}}{2\pi\mathrm{i}} \oint_C \mathrm{d}s\, \frac{s^{n+3/2}\mathrm{e}^{-2s}}{(s-\zeta)^{n+3/2}} = \frac{\mathrm{e}^{2\zeta}\zeta^{-1}}{\Gamma(n+3/2)} \frac{\mathrm{d}^{n+1/2}}{\mathrm{d}\zeta^{n+1/2}}(\zeta^{n+3/2}\mathrm{e}^{-2\zeta}) \tag{24}$$

where we have used the Osler–Nekrassov definition for the fractional derivative [22], $\Gamma(n)$ being the γ-function. To expand equation (24) we use the Leibniz rule generalization

$$\frac{\mathrm{d}^q[fg]}{\mathrm{d}\zeta^q} = \sum_{k=0}^{\infty} \binom{q}{k} \frac{\mathrm{d}^{q-k}f}{\mathrm{d}\zeta^{q-k}} \frac{\mathrm{d}^k g}{\mathrm{d}\zeta^k} \tag{25}$$

valid for arbitrary value of q, and the formulae [23]:

$$\frac{\mathrm{d}^{n+1/2-k}\zeta^{n+3/2}}{\mathrm{d}\zeta^{n+1/2-k}} = \frac{\Gamma(n+5/2)}{\Gamma(k+2)}\zeta^{k+1} \tag{26}$$

$$\frac{\mathrm{d}^k\mathrm{e}^{-2\zeta}}{\mathrm{d}x^k} = \frac{\mathrm{e}^{-2\zeta}}{\zeta^k}\gamma^*(-k, -2\zeta) \tag{27}$$

where $\gamma^*(c,\zeta)$ is the incomplete γ-function defined by [21]

$$\gamma^*(c,\zeta) = \frac{\zeta^{-c}}{\Gamma(c)} \int_0^{\zeta} t^{c-1} \mathrm{e}^{-t}\, \mathrm{d}t. \tag{28}$$

Substitution of the above expressions into equation (24) allows us to write $\psi_{1,n}(\zeta)$ as

$$\psi_{1,n}(\zeta) = 2\pi \frac{A'_{1,n}\zeta\, \mathrm{e}^{-2\zeta}}{\Gamma(n+\frac{3}{2})} \sum_{k=0}^{\infty} \binom{n+\frac{1}{2}}{k} \frac{\Gamma(n+\frac{5}{2})}{\Gamma(k+2)} \gamma^*(-k,-2\zeta). \tag{29}$$

5. Continuum-spectrum solutions

In the same spirit of study [18] on the 1D H atom, we perform an additional analysis of the $1/x$ potential in order to discern the possibility of having continuous spectrum solutions. In constrast to section 3 where we used a simpler integral transfer method but restricted ourselves to the condition that $\int_C^{\mathrm{d}} \mathrm{d}\bar{p}\, \phi_1(\bar{p})$ exist, in this section we use a less restrictive procedure which only requires the usual quadratic integrable condition. To this aim we substitute the expression $\psi_1(\zeta) = \int_{-\infty}^{\infty} \phi(\bar{p})\, \mathrm{d}\bar{p}\, \mathrm{e}^{\mathrm{i}\bar{p}\zeta}$ into equation (9) with $j=2$ to find

$$\int_C \mathrm{d}\bar{p} \left(\frac{\gamma}{\zeta} - \bar{p}^2 - 1\right) \phi(\bar{p})\mathrm{e}^{\mathrm{i}\bar{p}\zeta} = 0$$

which is an equivalent integral representation of the Schrödinger equation. Here C is the integration contourn. Now, using the identity $\mathrm{d}\,(\mathrm{e}^{\mathrm{i}\bar{p}\zeta}/\zeta)/\mathrm{d}\bar{p} = \mathrm{i}\mathrm{e}^{\mathrm{i}\bar{p}\zeta}$ for rewriting the second and third terms of this equation, yields

$$\frac{\gamma}{\zeta} \int_C \mathrm{d}\bar{p}\, \phi(\bar{p})\mathrm{e}^{\mathrm{i}\bar{p}\zeta} + \mathrm{i} \int_C \mathrm{d}\bar{p}\, (\bar{p}^2+1)\phi(\bar{p}) \frac{\mathrm{d}}{\mathrm{d}\zeta} \left[\frac{\mathrm{e}^{\mathrm{i}\bar{p}\zeta}}{\zeta}\right] = 0.$$

Integration of second term of this equation by parts leads to the expression

$$\frac{1}{\zeta} \int_C \mathrm{d}\bar{p} \left(\gamma\phi(\bar{p}) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\zeta}[(\bar{p}^2+1)\phi(\bar{p})]\right) \mathrm{e}^{\mathrm{i}\bar{p}\zeta} = 0 \tag{30}$$

provided

$$\frac{1}{\zeta} \Delta_C[(\bar{p}^2+1)\phi(\bar{p})\mathrm{e}^{\mathrm{i}\bar{p}\zeta}] = 0 \tag{31}$$

where Δ_C denotes the change in the value of the function between brackets in going from one end of the contourn to the other. To fulfill equation (30) we take for $\phi(\bar{p})$ the general solution of the differential equation obtained by making the brackets in equation (30) equal to zero. Then, substitution of the resulting solution in the inverse Fourier transform and in equation (31) yields

$$\psi_1(\zeta) = A'_2 \int_C \frac{\mathrm{e}^{-\mathrm{i}\gamma \arctan \bar{p}}}{\bar{p}^2+1} \mathrm{e}^{\mathrm{i}\bar{p}\zeta}\, \mathrm{d}\bar{p} \tag{32}$$

$$\frac{1}{\zeta} \Delta_C \left[\left(\frac{\mathrm{i}\bar{p}-1}{\mathrm{i}\bar{p}+1}\right)^{\gamma/2} \mathrm{e}^{\mathrm{i}\bar{p}\zeta}\right] = 0 \tag{33}$$

where A'_2 is an integration constant. The contourn C must be chosen to satisfy equation (33). If $\zeta \neq 0$ the possible contourn starts at infinity, surrounds the branch point $\bar{p} = \mathrm{i}$ and return to infinity so that the exponential vanishes on the positive part of the imaginary axis. Integration of equation (32) following this contourn leads to Whittaker functions [24] for which the value of γ is not restricted, so that the spectrum is continuous.

However, for $\zeta = 0$ this contourn does not comply with the condition given by equation (33) because it diverges. The only way to fulfill the condition for every ζ is to take a closed contourn along the real axis and to surround the point $\bar{p} = \mathrm{i}$, but now the function between the brackets of equation (33) has to be single valued in the upper-half part of the complex plane. This can only be satisfied if $\gamma/2$ is taken either as an integer or semi-integer. The semi-integer values of γ lead to the eigenfunctions found in section 4 and the integer ones yield to the same eigenfunctions of the 1D H atom equation (23) but without any absolute value in the variable ζ. As can be easily seen, these solutions diverge as $\zeta \to -\infty$, hence we ignored them because this behaviour is not physical.

Finally, let us compare the influence of the singularities of both potentials on their respective solutions. For this purpose, we integrate equation (9) around the origin to obtain

$$\frac{\mathrm{d}\psi_j\,(0^+)}{\mathrm{d}x} - \frac{\mathrm{d}\psi_j\,(0^-)}{\mathrm{d}x} + \frac{2m}{\hbar^2} \lim_{b\to 0} \int_{-b}^{b} \mathrm{d}x\, \left(\frac{Ze^2}{f_j(x)} - E \right) \psi_j(x) = 0. \tag{34}$$

For the 1D H atom ($j = 2$) the integral of equation (34) is finite for odd eigenfunctions ψ_j since for them $\psi_2(0) = 0$, but for even ones such that $\psi_2(0) \neq 0$, it diverges. Nevertheless equation (34) still may be satisfied if $\mathrm{d}\psi_2\,(0^+)/\mathrm{d}x = -\mathrm{d}\psi_2\,(0^-)/\mathrm{d}x = \infty$ and then the value of E turns out to be irrevelant because the term containing it remains finite; hence a continuum spectrum is the outcome. This is the case for the above-mentioned continuum-spectrum even solutions found by Heines and Roberts [16].

For the $1/x$ potential ($j = 1$), it is enough to ask $\psi_1(0)$ to remain finite in order to have a finite value for the integral of equation (34). However, for some solutions the slopes around the origin are divergent such that $\mathrm{d}\psi_1\,(0^+)/\mathrm{d}x = \mathrm{d}\psi_1\,(0^-)/\mathrm{d}x = \infty$ and $\psi_1(0) = 0$. This also leads to continuum-spectrum solutions such as the above-mentioned.

In summary, we have exactly obtained the 1D bound eigenenergies and eigenfunctions for two related potentials; one symmetric $(1/|x|)$ and the other antisymmetric $(1/x)$. For the former case our results reproduce those of Xianxi *et al* [5] but in a much simpler way. For the latter case, which to our knowledge has not been investigated before, we discussed the classical problem and performed the semiclassical treatment to find tunnelling through the increasingly infinite and narrow barrier at the origin. Finally, for the 1D $1/x$ potential we found a discrete spectrum, discussed the continuum spectrum solutions and gave analytical expressions of the eigenfunctions in terms of fractional derivatives which in turn can be written in terms of incomplete γ-functions. This solution is by itself important because it represents a simple 1D quantum mechanical system with an exact solution solved in momentum space. We hope that this paper may stimulate further work on the study of quantum problems with Coulomb-type and related singular potentials.

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